# Excited Coherent Modes of Ultracold Trapped Atoms

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## Abstract

A method of exciting coherent spatial modes of Bose–condensed trapped atoms is considered. The method is based on the resonance modulation of the trapping potential. The population dynamics of coherent modes is analysed. The method makes it possible to create mixtures of different spatial modes in arbitrary proportions, including the formation of pure excited coherent modes. Novel critical effects in the population dynamics are found.

# 1 Introduction

Bose atoms in magnetic traps can be cooled down to ultralow temperatures where Bose–Einstein condensation takes place. There exists numerous literature, both theoretical and experimental, devoted to this subject (see reviews [1,2]).

The Bose condensation occurs when the thermal wavelength of atoms, $\lambda \equiv \sqrt{2\pi\hbar^2/m_0 k_B T}$, becomes larger than the mean interparticle distance, $a$. At the same time, the effective radius of an atom, $a_0$, has to be much smaller than the interatomic distance in order that strong hard–core repulsion would not disturb much the motion of colliding particles. When the thermal wavelength is such that

$$\lambda \gg a \gg a_0 \, ,$$

atoms are mutually correlated and a coherent state develops.

In the system under equilibrium, condensed atoms are in the ground state. Recently [3], the idea was advanced of the possibility to create non–ground–state condensates corresponding to excited coherent modes of trapped atoms. This can be done by imposing, in addition to the trapping potential, a time–dependent field whose oscillation frequency is adjusted to be in resonance with the transition frequency between the ground–state level and a chosen excited level. In this report, we present the results of investigation of the population dynamics in a coherent system of trapped atoms, driven by a resonance field. As it turned out, the dynamics of such a system is quite nontrivial and is rather different from the population dynamics of two–level optical systems. Thus, unusual effects, that can be called critical, are found when a small variation of system parameters results in a drastic change of the population dynamics.

# 2 Population Dynamics

Consider the Bose gas of neutral atoms at ultralow temperatures, when all atoms are in a condensed state. The wave function of coherent atoms is obtained from the nonlinear Schrödinger equation

$$i\hbar \, \frac{\partial \varphi}{\partial t} = \left[ \hat{H}(\varphi) + V_{res} \right] \varphi \, , \tag{1}$$

in which $\hat{H}(\varphi)$ is a nonlinear Hamiltonian and $V_{res}$ is a resonance field. The gas is assumed to be diluted, and the interatomic interactions are shape independent and can be modelled by the contact potential

$$\Phi(\vec{r}) = \frac{4\pi\hbar^2 a_s}{m_0} \, \delta(\vec{r}) \, ,$$

where $m_0$ is atomic mass and $a_s$ is the scattering length. Then the nonlinear Hamiltonian is

$$\hat{H}(\varphi) = - \frac{\hbar^2}{2m_0} \, \vec{\nabla}^2 + U(\vec{r}) + A|\varphi|^2 \, , \tag{2}$$

where $U$ is a trapping potential and

$$A \equiv 4\pi\hbar^2 \, \frac{a_s}{m_0} \, N \, ,$$

represents the effective interaction for $N$ confined particles. The resonance field can be taken in the form

$$V_{res} = V(\vec{r}\,) \cos \omega t \, . \tag{3}$$

Let us note that the nonlinear Schrödinger equation is an *exact* equation of coherent states [4]. Contrary to this, if $\varphi$ is treated as the order parameter associated with the condensate, then Eq. (1) is an *approximate* equation corresponding to the mean–field approach at zero temperature. Such an approximate equation is often called the Gross–Ginzburg–Pitaevskii equation [5–7].

The stationary solutions of Eq. (1), if the time–dependent field is absent, are $\varphi_n \exp(-iE_n t/\hbar)$ with $\varphi_n$ and $E_n$ defined by the eigenvalue problem

$$\hat{H}(\varphi_n)\varphi_n = E_n\varphi_n \, . \tag{4}$$

The eigenfunctions $\varphi_n$, labelled by a multi–index $n$, are stationary *coherent modes*. The transition frequencies $\omega_{mn}$ between two energy levels are given by the difference

$$\hbar\omega_{mn} \equiv E_m - E_n \, . \tag{5}$$

Assuming that at the initial time $t = 0$, all atoms are in the ground–state coherent mode, we have the initial condition

$$\varphi(\vec{r}, 0) = \varphi_0(\vec{r}\,) \, . \tag{6}$$

And let then the resonant field (3) be switched on, with a frequency $\omega$ being in resonance, or almost in resonance, with the transition frequency

$$\omega_{j0} \equiv \frac{E_j - E_0}{\hbar} \tag{7}$$

between the ground state and a chosen energy level $j$. The corresponding quasiresonance condition is

$$\left| \frac{\Delta\omega}{\omega_{j0}} \right| \ll 1 \, , \qquad \Delta\omega \equiv \omega - \omega_{j0} \, . \tag{8}$$

The solution of the time–dependent equation (1) can be presented as an expansion

$$\varphi(\vec{r}, t) = \sum_n c_n(t)\varphi(\vec{r}\,) \, \exp\left(-\frac{i}{\hbar} \, E_n t\right) \tag{9}$$

over the coherent modes $\varphi_n(\vec{r}\,)$. Combining Eqs. (1) and (9), we find a system of equations for the coefficients $c_n(t)$. This system can be simplified in the resonance approximation giving

$$\frac{dc_0}{dt} = -i \, \alpha n_j c_0 - \frac{i}{2} \, \beta e^{i\Delta\omega t} \, c_j \, ,$$

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$$\frac{dc_j}{dt} = -i \, \alpha n_0 c_j - \frac{i}{2} \, \beta^* e^{-i\Delta\omega t} \, c_0 \, , \tag{10}$$

where

$$n_i(t) \equiv |c_i(t)|^2 \tag{11}$$

is the population of the level $i$, and a new notation is introduced for the interaction amplitude

$$\alpha \equiv \frac{A}{\hbar} \int |\varphi_0(\vec{r})|^2 |\varphi_j(\vec{r})|^2 \, d\vec{r} \tag{12}$$

as well as for the transition amplitude

$$\beta \equiv \frac{1}{\hbar} \int \varphi_0^*(\vec{r}) \, V(\vec{r}) \, \varphi_j(\vec{r}) \, d\vec{r} \, . \tag{13}$$

Because at the initial time all atoms are in the ground state, the initial conditions to Eqs. (10) are

$$c_0(0) = 1 \, , \qquad c_j(0) = 0 \, . \tag{14}$$

Since the quantities $c_0$ and $c_j$ are complex, Eqs. (10) are to be complemented by the equations for the complex conjugate $c_0^*$ and $c_j^*$ or by the equations for the populations (11). The latter equations are

$$\frac{dn_0}{dt} = \text{Im} \left( \beta e^{i\Delta\omega t} \, c_0^* c_j \right) \, ,$$

$$\frac{dn_j}{dt} = \text{Im} \left( \beta^* e^{-i\Delta\omega t} \, c_j^* c_0 \right) \, , \tag{15}$$

with the initial conditions

$$n_0(0) = 1 \, , \qquad n_j(0) = 1 \, . \tag{16}$$

In addition, the normalization condition

$$n_0(t) + n_j(t) = 1 \tag{17}$$

holds true. The derivation of Eqs. (10)–(17) was expounded in detail in Ref. [3].

The system of nonlinear differential equations (10) and (15) can be solved analytically by means of the averaging technique [8] yielding for the populations

$$n_0 = 1 - \frac{|\beta|^2}{\Omega^2} \, \sin^2 \frac{\Omega t}{2} \, , \qquad n_j = \frac{|\beta|^2}{\Omega^2} \, \sin^2 \frac{\Omega t}{2} \, , \tag{18}$$

in which $\Omega$ is an effective frequency given by the equation

$$\Omega^2 = [\alpha(n_0 - n_j) - \Delta\omega]^2 + |\beta|^2 \, . \tag{19}$$

The quantity $\Omega$ is an equivalent for the Rabi frequency, although one has to keep in mind that it is actually not a fixed frequency but a function of time defined by

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Eqs. (18) and (19). The solutions (18), for $|\beta| < \alpha$, give a qualitative picture of complicated nonlinear oscillations of the populations (11). In the limit $|\beta| \ll \alpha$, these solutions become asymptotically exact.

Equations (15) show that if in some moment of time the pumping resonance field (3) is switched off, then after this the populations $n_0$ and $n_j$ remain constant at their instantaneous values. This suggests the way of creating mixtures of coherent modes with different spatial configurations, and even pure excited coherent modes. Such excited coherent modes will not, of course, last for ever. Their lifetime is defined by the total lost rate $\gamma_j$ caused by atomic collisions. The same, actually, concerns the ground–state condensate whose lifetime is defined by the corresponding lost rate $\gamma_0$. The lost rates due to binary collisions can be presented as

$$\gamma_j = \lambda_{jj} N^2 n_j^2 \int |\varphi_j(\vec{r}\,)|^4 \, d\vec{r} + \lambda_{j0} N^2 n_j n_0 \int |\varphi_j(\vec{r}\,)|^2 |\varphi_0(\vec{r}\,)|^2 \, d\vec{r} \,,$$

$$\gamma_0 = \lambda_{00} N^2 n_0^2 \int |\varphi_0(\vec{r}\,)|^4 \, d\vec{r} + \lambda_{0j} N^2 n_0 n_j \int |\varphi_0(\vec{r}\,)|^2 |\varphi_j(\vec{r}\,)|^2 \, d\vec{r} \,, \qquad (20)$$

where $\lambda_{ij}$ are the corresponding relaxation coefficients. The lost rates due to ternary collisions can be presented in the similar way. The ternary loss rates are usually much smaller than the binary ones [9].

The approximate solutions (18) and (19), obtained by the averaging method, give us a general understanding of the excitation procedure. However, Eqs. (18) and (19) define the level populations not explicitly but through a connected system of equations. In order to study the dynamics of the populations (11) explicitly, and also not to be limited by the averaging–technique approximation, we return back to the evolution equations (10), which we solve numerically. For numerical analysis, it is convenient to measure time in units of $\alpha^{-1}$ and to introduce dimensionless quantities

$$b \equiv \frac{|\beta|}{\alpha} \,, \qquad \delta \equiv \frac{\Delta\omega}{\alpha} \,.$$

The results of numerical calculations for several values of parameters are presented in Figs. 1 to 6.

It turned out, that the dynamics of the populations is rather nontrivial and exhibits a kind of critical effects when an anomalous coherent oscillation of the state populations suddenly appears. The latter occur on the critical line defined approximately as $b_c = 0.5 - \delta$. Outside the critical line, the populations oscillate according to the sine–squared law, as in Fig.1. Approaching the critical line, for instance by changing the detuning, the period of oscillations suddenly increases by a jump, with the top of $n_j$ and, respectively, the bottom of $n_0$, becoming flat. Thus, the slight change of the detuning, from $\delta = 0.11$ in Fig.1 to $\delta = 0.11017478$ in Fig.2, results in an abrupt period doubling. And the following small shift of the detuning to $\delta = 0.11017479$ results again in the period being approximately doubled, with the qualitative change of the time–dependence: Each second downward cusp between two adjacent oscillations of $n_j$ overturns up becoming an upward cusp,

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which yields to an effective period doubling, as is seen from comparing Figs. 2 and 3. The further increase of the detuning drives the system away from the critical line resulting in the decrease of the oscillation period and in the oscillation behaviour again more resembling the sine–squared shape, as in Figs. 4 and 5. The overall picture is qualitatively the same, when we cross the critical line at other values of the parameters $b$ and $\delta$. Thus, Fig.6 shows the time behaviour of the populations at $b = 0.499$ and $\delta = 0.001001002$ being on the critical line. Figure 6 is similar to Fig. 2, except that the oscillation amplitudes are different, being equal to different values of $b$.

In practice, the value of $b$, which is the dimensionless expression for the transition amplitude (13), depends on the resonant field (3). The spatial part of the latter can be chosen to have different forms. For example, in the consideration of a mixture of two Bose condensates of $^{87}$Rb atoms in two internal hyperfine states, en effective potential forcing the excitation of a first antisymmetric mode was linear [10], due to the spatial separation of the condensate components. In our scheme, the potential $V(\vec{r})$ can be arbitrary.

# 3   Conclusion

We presented the analysis of temporal behaviour of spatial coherent modes excited by a resonant field realizing the oscillatory modulation of the trapping potential. Such a procedure of using the resonant pumping field suggests the way of creating mixtures of coherent spatial modes in arbitrary proportions, including the formation of pure excited coherent modes.

There exists the critical line connecting the values of the transition amplitude and detuning at which the dynamics of fractional populations suddenly changes. The qualitative and sharp change in the population dynamics reminds critical phenomena occurring in equilibrium systems. It is possible to construct an effective stationary system describing the averaged behaviour of populations satisfying the original evolution equations. The effective averaged system displays critical behaviour at the critical line which corresponds to that observed for the nonequilibrium system. Effective critical indices can also be defined. The detailed study of the critical dynamics is presently under investigation and will be published elsewhere. The possibility of exciting various coherent modes of Bose atoms can be important in the context of realizing different spatial modes of atom lasers [11–15].

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# Figure captions

**Fig.1.** The populations of the excited coherent mode (solid line) and of the ground–state mode (dashed line) as functions of time for $b = 0.4$ and $\delta = 0.11$.

**Fig.2.** The critical dynamics of the populations for $b = 0.4$ and $\delta = 0.11017478$.

**Fig.3.** Overturning of the downward cusps of $n_j$ upward, with the period of oscillations being approximately doubled, occuring at $b = 0.4$ and $\delta = 0.11017479$.

**Fig.4.** The time dependence of the populations outside the critical line, with $b = 0.4$ and $\delta = 0.12$.

**Fig.5.** The fractional population oscillations for $b = 0.4$ and $\delta = 0.2$.

**Fig.6.** The critical population dynamics for $b = 0.499$ and $\delta = 0.001001002$. As in all previous figures, solid line corresponds to $n_j$ and the dashed one to $n_0$.